Controlled Invariance and Dynamic Feedback for Systems over Semirings

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Abstract

The concept of (A, B)-invariant subspace is the fundamental concept of the geometric approach of control design. It has been extended by many authors to that of (A, B)-invariant module or semimodule, for the sake of extending the solution of various control problems to the case of systems over rings or semi rings. In this paper is discussed the use of dynamic feedback control laws for systems over semirings, and it is shown that an (A, B)-invariant semimodule over a commutative semiring can be made invariant for the closed-loop system by dynamic feedback.

1 Introduction

The concept of (A, B)-invariance, also called controlled invariance, was introduced independently in [1] and [2]. It constitutes the basic stone of the geometric approach to the control theory of linear dynamical systems, that provided solutions to many control problems, among which are the disturbance decoupling problem, the regulator problem, the model matching problem, to name but a few of them, as presented in the seminal references [3, 4]. This motivated many authors to consider the extension of this approach to the theory of linear dynamical systems over rings [5, 6] and semirings [7].

In the case of linear dynamical systems with coefficients in a field, it is well know that the controlled invariance of a subspace is equivalent to its invariance for the closed-loop system obtained by the action of a state feedback, (see [3, 4]). This property makes the (A, B)-invariant spaces very useful in the classical theory. Unfortunately, this feature so important is generally lost in the framework of linear systems with coefficients in a ring or a semiring. Although feedback invariance always implies controlled invariance, the converse does not hold true in general (see [5, 7]).

In the search to establish somehow an equivalence between these notions of invariance, Conte and Perdon introduced in [6] a notion of dynamic feedback invariant submodules for linear systems over a principal ideal domain, which is a generalized notion of the (static) feedback invariance, and showed that controlled invariance is equivalent to dynamic feedback invariance. These results were extended by Ito et al. in [8] to the case of linear systems over a commutative Noetherian domain. The method was based on the construction of an extended system. It permitted to solve the disturbance decoupling problem with measurable disturbance by dynamic state feedback in [6]. Di Loreto *et al.* also used dynamic feedback in [9] for the sake of analyzing duality for systems over a commutative ring.

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For systems over a semiring, Katz establishes in [7] that controlled invariance is equivalent to invariance after static state feedback, in the case of a semimodule with finite volume. Other particular cases are implicitly treated in [11, 12], where is studied the control of max-plus linear systems subject to strict temporal constraints, that is an example of control problem in this framework that comes down to the controlled invariance of a semimodule. It is interesting to mention that Katz indirectly utilizes the notion of dynamic feedback in an example of transportation networks subject to timed constraints. He first introduces an extended system to describe the prescribed constraints in the form of the controlled invariance of a semimodule, then the maximal controlled invariant semimodule is made feedback invariant by a static state feedback on the extended system.

Inspired by the theory of systems over ring, the purpose of the present paper is to show that, for systems over semirings, every controlled invariant semimodule can be made invariant using a dynamic feedback. For this, a new definition of dynamic feedback type invariance is introduced. This result can be useful to resolve control design problems and extend the geometric approach to systems over semirings.

This paper is organized as follows. In Section 2, some backgrounds on semirings are recalled. In Section 3, we recall the concepts of controlled invariance and controlled invariance of feedback type. Then the concept of dynamic feedback is introduced and the equivalence between controlled invariance and controlled invariance of dynamic feedback type is shown in Section 4. Two examples taken from the literature illustrate the obtained result in Section 5, and concluding remarks are given in Section 6.

2 Preliminaries

A monoid is a set endowed with an internal operation that is associative and has an identity element. A semiring is a set $\mathcal{S}$ endowed with two operations, denoted $\oplus$ and $\otimes$, respectivelly called addition and multiplication, that satisfy the following axioms. $(\mathcal{S}, \oplus)$ is a commutative monoid, $(\mathcal{S}, \otimes)$ is a monoid, the multiplication $\otimes$ is distributive over finite sums (say, $\forall a, b, c \in \mathcal{S}, \quad (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$, and $c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$,) and the neutral element of the addition is absorbing with respect to the multiplication. The neutral element of the multiplication is denoted e, and called unity, and the neutral element of the addition is denoted ϵ, and is said to be null.

Classical examples of semirings are the set of natural integers $\{\mathbb{N}, +, \times\}$, $\{\mathbb{R} \cup \{-\infty\}, \max, +\}$, that is sometimes called max-plus algebra, and $\{\mathbb{N} \cup \{+\infty\}, \min, +\}$, called tropical algebra. Many other examples and applications are for instance provided in [13].

The notations $\oplus$ and $\otimes$ are extended to vectors and matrices, as usually. For $p, q \in \mathbb{N}$, we denote by $\mathcal{S}^{p \times q}$ the set of all matrices of order $p \times q$ with coefficients in the semiring $\mathcal{S}$. Two matrices $A, B \in \mathcal{S}^{m \times n}$ being given, we define the sum $A \oplus B$ by $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$, and with $A \in \mathcal{S}^{p \times n}$, $B \in \mathcal{S}^{n \times m}$, we define the product $A \otimes B$ by $(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}$. The product $A \otimes B$ is sometimes written AB, and the null matrix is denoted ϵ, when no confusion can arise. The unit matrix I_n is the matrix of size $n \times n$ which entries equal e on diagonal elements, and ϵ elsewhere.

The analogues of vector spaces or modules, obtained by replacing the field or ring of scalars by an idempotent semiring, are called semimodules. Here, we will consider only subsemimodules of the Cartesian product $\mathcal{S}^n$.

If C is a $n \times m$ matrix over $\mathcal{S}$, we denote by $\operatorname{Im} C$ the subsemimodule of $\mathcal{S}^n$ generated by the columns of C. If a subsemimodule $\mathcal{C}$ of $\mathcal{S}^n$, can be expressed as $\mathcal{C} = \operatorname{Im} C$ for some matrix $C \in \mathcal{S}^{n \times p}$ and some finite integer p, it is said that the subsemimodule $\mathcal{C}$ is finitely generated.

By a linear system over the semiring $\mathcal{S}$, we mean a linear dynamical system whose evolution is determined by a set of equations of the form

$$x(k+1) = A \otimes x(k) \oplus B \otimes u(k+1) \ , \tag{2.1}$$

where $A \in \mathcal{S}^{n\times n}$, $B \in \mathcal{S}^{n\times q}$, $x(k) \in \mathcal{S}^n$ is the state vector, and $u(k) \in \mathcal{S}^m$ is the control input vector, both defined for $k \in \mathbb{N}$.

3 Backgrounds on controlled invariance

In this section, let us first recall some definitions and results of invariance over semirings [7].

Definition 3.1. (Controlled invariance) Given $A \in \mathcal{S}^{n\times n}$ and $B \in \mathcal{S}^{n\times q}$, a semimodule $\mathcal{M} \in \mathcal{S}^n$ is said to be (A, B)-invariant, or controlled invariant, if

$$A\mathcal{M} \subset \mathcal{M} \ominus \operatorname{Im} B, \tag{3.2}$$

where $\mathcal{M} \ominus \operatorname{Im} B$ is defined as the set $\{x \in \mathcal{S}^n \mid \exists b \in \operatorname{Im} B, x \oplus b \in \mathcal{M}\}$.

Definition 3.2. Given $A \in \mathcal{S}^{n\times n}$ and $B \in \mathcal{S}^{n\times q}$, a semimodule $\mathcal{M} \in \mathcal{S}^n$ is said to be (A, B)-invariant of feedback type, or controlled invariant of feedback type, if there exists a matrix $F \in \mathcal{S}^{q\times n}$ such that

$$(A \oplus BF)\mathcal{M} \subset \mathcal{M}. \tag{3.3}$$

A number of observations can be done regarding these concepts. They are now recalled without proof, see for instance [7] for details.

Theorem 3.1. The following properties are met.
(i) A semimodule $\mathcal{M}$ is (A, B)-invariant if and only if for every initial condition $x(0) \in \mathcal{M}$, there exists a control law $u(k)$, defined for $k \geq 1$, such that the state remains in $\mathcal{M}$ for the whole evolution of the system: $x(k) \in \mathcal{M}$, for $k \geq 0$.
(ii) A semimodule $\mathcal{M}$ is (A, B)-invariant of feedback type if and only if there exists a matrix $F \in \mathcal{S}^{q\times n}$ such that each trajectory of the closed loop system, $x(k) = (A \oplus BF)x(k-1)$, is completely contained in $\mathcal{M}$ when its initial state is in $\mathcal{M}$.
(iii) As a consequence, every (A, B)-invariant semimodule of feedback type is also (A, B)-invariant. The converse assertion is not true, in general.
(iv) A finitely generated semimodule $\mathcal{M} \subset \mathcal{S}^n$, generated by a matrix $M \in \mathcal{S}^{n\times p}$, is (A, B)-invariant if and only if there exist matrices $X \in \mathcal{S}^{q\times p}$, $Y \in \mathcal{S}^{p\times p}$ such that the following equality holds true :

$$A \otimes M \oplus B \otimes X = M \otimes Y \ . \tag{3.4}$$

(v) A finitely generated semimodule $\mathcal{M} \subset \mathcal{S}^n$, generated by a matrix $M \in \mathcal{S}^{n\times p}$, is (A, B)-invariant of feedback type if and only if there exist matrices $F \in \mathcal{S}^{q\times n}$ and $Y \in \mathcal{S}^{p\times p}$, such that the following equality holds true :

$$(A \oplus B \otimes F) \otimes M = M \otimes Y \ . \tag{3.5}$$

Assertions (i) and (ii) are behavioral properties that help understanding the real meaning of controlled invariance. Properties (iv) and (v) are algebraic characterizations, that are useful in practice to check the controlled invariance of finitely generated semimodules. The main obstacle to apply the concept of (A, B)-invariance to the solution of control design problems lies in the assertion (iii) of the above remark.

In many problems, one can show the existence of a maximal (A, B)-invariant subsemimodule of any semimodule, but in general one cannot calculate a static state feedback F that makes it invariant for the closed-loop system, even in the case of a finitely generated semimodule. This comes from the fact that X cannot be factorized in the form $X = FM$, in general. In the same way, there is no maximal (A, B)-invariant subsemimodule of feedback type included in a given semimodule, in general, which, together with the fact that characterization (3.5) is nonlinear, makes difficult the computation of a control law that keeps the trajectories of the system in the given semimodule. The concept of dynamic feedback resolves this problem.

4 Invariance by dynamic feedback

Let us now introduce a new notion of dynamic feedback for systems over a semiring $\mathcal{S}$, and of (A, B)-invariance of dynamic feedback type.

Definition 4.1. (Dynamic feedback) A dynamic feedback is a control law of the form

$$u(k) = Ex(k) \oplus Fz(k) \ , \tag{4.6}$$

for $k \geq 1$, where E and F are matrices of convenient sizes, and $z(k) \in \mathcal{S}^q$ is an internal variable of the controller, which evolution is directed by

$$z(k+1) = Gx(k) \oplus Hz(k) \ , \tag{4.7}$$

with matrices G and H of convenient size.

Notice that to define the control law in a unique way, Eq. (4.7) needs an initialization. We will come back to this question at the end of the section, when summarizing the controller design method. The dynamic feedback defined by (4.6-4.7) gives rise to the closed-loop system

$$\begin{pmatrix} x(k+1) \\ z(k+1) \end{pmatrix} = \begin{pmatrix} A \oplus BE & BF \\ G & H \end{pmatrix} \begin{pmatrix} x(k) \\ z(k) \end{pmatrix} ,$$

that coincides with the closed-loop system obtained applying the extended static state feedback

$$\begin{pmatrix} u(k+1) \\ w(k+1) \end{pmatrix} = F_e \begin{pmatrix} x(k) \\ z(k) \end{pmatrix}$$

to the open-loop extended system

$$\begin{pmatrix} x(k+1) \\ z(k+1) \end{pmatrix} = A_e \begin{pmatrix} x(k) \\ z(k) \end{pmatrix} \oplus B_e \begin{pmatrix} u(k+1) \\ w(k+1) \end{pmatrix} ,$$

with

$$A_e = \begin{pmatrix} A & \epsilon \\ \epsilon & \epsilon \end{pmatrix}, \ B_e = \begin{pmatrix} B & \epsilon \\ \epsilon & I_p \end{pmatrix} ,$$

and

$$F_e = \begin{pmatrix} E & F \\ G & H \end{pmatrix} . \tag{4.8}$$

We introduce now the concept of extended semimodule of a finitely generated semimodule.

Definition 4.2. A semimodule $\mathcal{M} \subset \mathcal{S}^n$ being given, generated by the matrix $M \in \mathcal{S}^{n\times p}$, the extension of $\mathcal{M}$, denoted $\mathcal{M}_e$, is the semimodule of $\mathcal{S}^{n+p}$ generated by the concatenated matrix M_e defined by

$$M_e = \begin{pmatrix} M \\ I_p \end{pmatrix} .$$

We can now state our main result

Theorem 4.1. Be given the system (2.1), let $\mathcal{M}$ be a finitely generated semimodule of $\mathcal{S}^n$, the following assertions are equivalent.
(i) $\mathcal{M}$ is (A, B)-invariant.
(ii) $\mathcal{M}_e$ is (A_e, B_e)-invariant.
(iii) $\mathcal{M}_e$ is (A_e, B_e)-invariant of feedback type.
(iv) There exists matrices E, F, G, H such that, for every initial value $x(0)$, the state of the closed-loop system formed by the interconnection of (2.1) and of the dynamic state feedback of the form (4.6)-(4.7), with $q = p$ and the initial value $z(0)$ chosen so that $x(0) = Mz(0)$, satisfies $x(k) \in \mathcal{M}$, for every $k \geq 0$.

Proof Since $\mathcal{M}$ is a finitely generated semimodule, there exists a matrix $M \in \mathcal{S}^{n\times p}$, for some $p \in \mathbb{N}$, such that $\mathcal{M} = \operatorname{Im} M$. We first prove that (i) implies (ii) and (iii). Assume that $\mathcal{M}$ is controlled invariant. It follows, by (3.4), that there exist matrices $X \in \mathcal{S}^{n\times p}$ and $Y \in \mathcal{S}^{p\times p}$ such that the equality $A \otimes M \oplus B \otimes X = M \otimes Y$ holds true.

Then, it can be seen that the following matrix equality holds true too,

$$\begin{pmatrix} M \\ I_p \end{pmatrix} Y = (A_e \oplus B_e F_e) \begin{pmatrix} M \\ I_p \end{pmatrix} , \tag{4.9}$$

taking in (4.8) $E = \epsilon$, $F = X$, $G = \epsilon$, and $H = Y$, so that we now have

$$F_e = \begin{pmatrix} \epsilon & X \\ \epsilon & Y \end{pmatrix} . \tag{4.10}$$

It shows that the image of the concatenated matrix $(M^T, I_p)^T$ is an (A_e, B_e)-invariant of feedback type for the closed-loop system. This establishes that (i) implies (iii). From assertion (iii) of Theorem 3.1, it is clear that (iii) in turn implies (ii). We can now show that reversely (ii) implies (iii). To this aim, we first remark that assertion (ii) implies the existence of matrices $X_e = (X^T, Y^T)^T$, and Y_e, such that the equality

$$A_e M_e \oplus B_e X_e = M_e Y e .$$

Further, from the definition of M_e, one can factorize X_e on the form $X_e = F_e M_e$, taking like previously F_e defined by (4.10). One finally observe that $(A_e \oplus B_e F_e)M_e = M_e Y_e$, which establishes the implication. It remains to show that (iii) implies (i). This is obtained writing that if there exists F_e (not necessarily in the form (4.8)) and Y_e such that equality $(A_e \oplus B_e F_e)M_e = M_e Y_e$ is fulfilled, then one observes that $AM \oplus BX = MY$, taking $Y = Y_e$ and $X = EM \oplus F$, where the matrices E and F are obtained partitioning F_e in the form (4.8).

To prove that (iv) implies (i), we first remark that since $\mathcal{M}$ is generated by the columns of M, for each vector $x(0) \in \mathcal{M}$, there exists a vector $v \in \mathcal{S}^p$ such that $x(0) = Mv$. We further remark that assertion (iv) implies that there exists matrices E and F and, for every $x(0) \in \mathcal{M}$, vectors v satisfying $x(0) = Mv$ and $x(1) \in \mathcal{M}$, such that $(A \oplus BE)x(0) \oplus BFv$. Taking successively the different columns of M for initial condition $x(0)$, one defines matrices $Y \in \mathcal{S}^{p\times p}$ and $V \in \mathcal{S}^{p\times p}$, formed by the successive values obtained for v and $x(1)$, that satisfy $MY = (A \oplus BE)M \oplus BFV$. Taking finally $X = EM \oplus FV$, it appears that (3.4) is satisfied, that shows that $\mathcal{M}$ is (A, B)-invariant.

To complete the proof, we show that (i) implies (iv). In this part of the proof is discussed for the first time the initialization of the dynamic feedback, that is important for the real implementation of such a control law. We already noticed that the existence of matrices X and Y satisfying (3.4) leads to the definition of the extended feedback F_e as in (4.10), that is such that $(A_e \oplus B_eF_e)M_e = M_eY$. This actually implies that $(A_e \oplus B_eF_e)^kM_e = M_eY^k$, for $k \geq 1$, so that for every vector $v \in \mathcal{S}^p$, one obtains $(A_e \oplus B_eF_e)^kM_ev = M_eY^kv$, for $k \geq 1$. Since $M_e = (M^T, I_p)^T$ and the solution of the closed-loop system obtained by the action of the dynamic feedback that corresponds to F_e is given by $(x^T(k), z^T(k))^T = (A_e \oplus B_eF_e)^kM_e(x^T(0), z^T(0))^T$, we verify that taking $z(0) = v$, where $Mv = x(0)$, the equality $x(k) = MY^kv$ holds true, and that therefore $x(k) \in \mathcal{M}$, for $k \geq 1$. This completes the proof.

As recalled in Theorem 3.1, the controlled invariance of a semimodule means that the trajectories of the system can be forced by control to stay in the semimodule during their evolution. Theorem 4.1 shows that in the case of a finitely generated semimodule, a control that forces the trajectories to stay in the given semimodule can be realized using a dynamic feedback. Such a control law is causal, which permits its real implementation. The following formulation summarizes the design method suggested in the proof of Theorem 4.1.

Corollary 4.1. Be given a controlled invariant semimodule $\mathcal{M} \subset \mathcal{S}^n$, a control law that forces $x(k)$ to stay in $\mathcal{M}$, $\forall k \geq 0$, if the initial condition $x(0)$ already is in $\mathcal{M}$, is provided by

$$u(k+1) = Xz(k) \ , \tag{4.11}$$

for $k \geq 0$, where $z(k)$ is defined by

$$z(k+1) = Yz(k) \ , \tag{4.12}$$

for $k \geq 0$, and

$$z(0) = v \ , \tag{4.13}$$

the matrices X and Y being solutions of the equation

$$MY = AM \oplus BX \ , \tag{4.14}$$

and v solution of

$$x(0) = Mv \ . \tag{4.15}$$

The starting entries of the control design are hence the knowledge of a generating matrix M, of the model of the system (A, B), and of the initial state $x(0)$. The solution of Eqs. (4.14) and (4.15) is the key to calculate the control law parameters. The identities (4.11) and (4.12) are implemented to realize the online computation of the control, and (4.13) is used at initial time to initialize the controller.

The proposed method is effective for the rings and semirings for which algorithms for solving equations of the form (4.14) and (4.15) has been described. Such algorithms are discussed in many publications. The real implementation of this design method will be the aim of further work. We now present a formal example.

5 Illustrative examples

Example 1: We consider a system of the form

$$x(k+1) = Ax(k) \oplus Bu(k+1) ,$$

over the semiring $\mathcal{D} = \{\mathbb{N} \cup \{+\infty\}, \oplus, \otimes\}$, where $\oplus$ denotes the operation min, and $\otimes$ the usual addition, and

$$A = \begin{pmatrix} 1 & +\infty \\ 1 & 0 \end{pmatrix} , \; B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

The semimodule $\mathcal{K}$ is defined as $\mathcal{K} = \{(x^T, y^T)^T \in \mathcal{D}^2 \,|\, x \leq y\}$. This example is proposed by Katz [7], who shows that the maximal (A, B)-invariant semimodule included in $\mathcal{K}$ is well-defined and given by $\mathcal{K}^\star = \{(x^T, y^T)^T \in \mathcal{D}^2 \,|\, x \leq y \text{ and } 1 \leq y\}$. The author remarks that $\mathcal{K}^\star$ is not an (A, B)-invariant semimodule of feedback type. Accordingly to this remark, there is no matrix F over $\mathcal{D}$ such that $\mathcal{K}^\star$ is $(A + BF)$-invariant. Katz also remarks for this example that the constant control $u(k) = 0$ makes the state $x(k)$ staying in $\mathcal{K}^\star$ if its initial value $x(0)$ already lies in this semimodule. According to the previous results, there also exists a dynamic feedback that forces the trajectory $x(k)$ to stay in $\mathcal{K}^\star$. We can calculate it using the given formulae, in the form $u(k+1) = X \otimes z(k)$, where $z(k)$ is defined by the recurrence $z(k+1) = Y \otimes z(k)$, initialized to any value $z(0)$ satisfying $Mz(0) = x(0)$, and where X and Y are solutions of $A \otimes M \oplus B \otimes X = M \otimes Y$, and M is a generating matrix of the semimodule $\mathcal{K}^\star$. One can take for instance

$$M = \begin{pmatrix} 1 & 0 \\ 1 & +\infty \end{pmatrix} , \; v = \begin{pmatrix} \max\{x_1 - 1, x_2 - 1\} \\ x_1 \end{pmatrix} ,$$

where x_1 and x_2 are the components of the initial state, say $x(0) = (x_1^T, x_2^T)$, and

$$X = \begin{pmatrix} 0 & 0 \end{pmatrix} , \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} .$$

Example 2: Consider the example from Maia et al treated in [11],

$$x(k+1) = Ax(k) \oplus Bu(k+1) , \tag{5.16}$$

over the semiring max plus: $\mathbb{R}_{\max} = \{\mathbb{R} \cup \{-\infty\}, \oplus, \otimes\}$, where $\oplus$ denotes the operation max, and $\otimes$ the usual addition, and

$$A = \begin{pmatrix} 0 & -\infty & 5 & -\infty \\ 10 & 0 & 15 & 7 \\ 4 & -\infty & 9 & -\infty \\ 15 & 5 & 20 & 12 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & -\infty & -\infty & -\infty \\ 10 & 0 & -\infty & -\infty \\ 4 & -\infty & 0 & -\infty \\ 15 & 5 & -\infty & 0 \end{pmatrix} .$$

This example is a model of a simple traffic light. The problem consists in satisfying time synchronization constraints described by the semimodule $\mathcal{D} = \operatorname{Im} D$, where

$$D = \begin{pmatrix} 0 & -15 & -15 & -30 \\ 10 & 0 & -5 & -15 \\ 6 & -11 & 0 & -26 \\ 15 & 5 & 0 & 0 \end{pmatrix} .$$

A similar time constraints problem can be found in [7], where the goal is to maintain the system trajectory within a given semimodule that expresses time table synchronization constraints of train departures.

Since the semimodule $\mathcal{D}$ is finitely generated, taking into account Theorem 3.1, the semimodule $\mathcal{D}$ is (A,B)-invariant if there exist matrices $X, Y \in \mathbb{R}_{\max}^{4\times 4}$ such that the equality $AM \oplus BX = MY$, holds true.

In fact, consider X and Y as follows,

$$X = \begin{pmatrix} -\infty & 0 & -\infty & 0 \\ -\infty & -\infty & -\infty & -\infty \\ 17 & 6 & 11 & 6 \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}$$

$$Y = \begin{pmatrix} 11 & 0 & 5 & 0 \\ 22 & 12 & -\infty & -\infty \\ 15 & -\infty & -\infty & -\infty \\ 19 & -\infty & -\infty & -\infty \end{pmatrix}.$$

Let us now see that $\mathcal{D}$ is not invariant of feedback type. With this aim, one can show that a trajectory which starts at the point $x_0 = (-15 \quad -5 \quad 0 \quad 0)^T \in \mathcal{D}$ cannot be kept inside $\mathcal{D}$ when a linear state feedback is applied. Let $F \in \mathbb{R}_{\max}^{4\times 4}$ be an arbitrary feedback, then $(A \oplus BF)x_0 \in cl\{(5\ 15\ 9\ 20)^T, (0\ 10\ 4\ 15)^T,$
$(\epsilon\ 0\ \epsilon\ 5)^T, (\epsilon\ \epsilon\ 0\ \epsilon)^T, (\epsilon\ \epsilon\ \epsilon\ 0)^T\}$, and it is clear that the vectors that form the linear combination of $(A \oplus BF)x_0$ are not in $\mathcal{D}$, thus $(A \oplus BF)x_0 \notin \mathcal{D}$. Therefore, it is not possible to apply a static control law such that the trajectory $x(k)$ of the system, satisfies the imposed constraints.

Now, according to Corollary 4.1, there exists a dynamic feedback that forces the trajectory $x(k)$ to stay in $\mathcal{D}$.

Let us consider the evolution of the system (5.16) when the inicial state is $x(0)=(x_1\ x_2\ x_3\ x_4)^T \in \mathcal{D}$, and the dynamic control law defined by

$$u(k+1) = \begin{pmatrix} -\infty & 0 & -\infty & 0 \\ -\infty & -\infty & -\infty & -\infty \\ 17 & 6 & 11 & 6 \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix} \otimes z(k)\ ,$$

for $k \geq 0$, where $z(k)$ is defined by

$$z(k+1) = \begin{pmatrix} 11 & 0 & 5 & 0 \\ 22 & 12 & -\infty & -\infty \\ 15 & -\infty & -\infty & -\infty \\ 19 & -\infty & -\infty & -\infty \end{pmatrix} \otimes z(k)\ ,$$

for $k \geq 0$, is applied.

For instance, one can take

$$z(0)=\begin{pmatrix} \min\{x_4 - 15,\, x_3 - 6,\, x_2 - 10,\, x_1\} \\ \min\{x_4 - 5,\, x_3 + 11,\, x_2,\, x_1 + 15\} \\ x_4 \\ \min\{x_4,\, x_3 + 26,\, x_2 + 15,\, x_1 + 30\} \end{pmatrix},$$

Let us consider for example, the inicial state $x(0) = (-15\ \ -5\ 0\ 0)^T \in \mathcal{D}$, then $z(0) = x(0)$ in this case, and we obtain the following trajectory $x(k)$ of the system

$$\left\{ \begin{pmatrix} 5 \\ 15 \\ 11 \\ 20 \end{pmatrix}, \begin{pmatrix} 16 \\ 27 \\ 22 \\ 32 \end{pmatrix}, \begin{pmatrix} 27 \\ 39 \\ 33 \\ 44 \end{pmatrix}, \begin{pmatrix} 39 \\ 51 \\ 45 \\ 56 \end{pmatrix}, \ldots \right\}$$

for the sequence of control vectors
$u(1)=Xz(0)=(0\ \epsilon\ 11\ \epsilon)^T, u(2)=Xz(1)=(7\ \epsilon\ 22\ \epsilon)^T, u(3) = Xz(2) = (27\ \epsilon\ 33\ \epsilon)^T, u(4) = Xz(3) = (39\ \epsilon\ 45\ \epsilon), \cdots$,
where
$z(1) = Yz(0) = (5\ 7\ 0\ 4)^T, z(2) = Yz(1) = (16\ 27\ 20\ 24)^T, z(3) = Yz(2) = (27\ 39\ 31\ 35)^T, \cdots$,

Clearly, the trajectory $x(k)$ of the system satisfies the imposed constraints, when a dynamic control law is applied.

6 Conclusions

In this paper, the concept of dynamic feedback for (A,B)-invariant semimodules of linear systems over semirings is presented. It was shown that closed-loop invariance under dynamic feedback is equivalent to controlled invariance. This closes a question which was open, that of the causal realization of a control law for a finitely generated (A, B)-invariant system over a commutative semiring.

Since a ring can be seen as a particular instance of the concept of semiring, the result also generalizes the results that have been described for systems over rings. The existence of a dynamic feedback making invariant an (A, B)-invariant module has indeed been proved in the case of systems over a principal ideal domain [6], and of systems over a noetherian ring [8]. Our construction of a dynamic feedback control law is actually based on the one suggested in [9], that was for systems over a commutative ring. In the present contribution, the interpretation in terms of causal control of this construction is emphasized.

In the case of systems over a semiring, the question was pointed out by Katz [7]. It is interesting to point out that in the examples of this paper, were already considered causal and dynamical feedbacks to solve the invariance problem. In the example of its section VI, a dynamic feedback is actually used. In this case, its computation is done using the main theorem of the paper, that is a design method for a static state feedback, in the particular case of a semimodule with finite volume. The static state feedback actually comes down to a dynamic feedback, since the control specification is expressed in terms of a temporal constraint, that includes a delayed term, that is seen as a static constraint on an extended space. In the same paper, Example 4 (that we discussed in our section 4) is given as an instance of a semimodule that is (A, B)-invariant but is not feedback invariant. The author however mentions the existence of a constant control that makes invariant the semimodule for the controlled system. We just point out that such a control law is no longer a linear feedback, but is causal. In this sense, the results of the paper generalize these premises.

The results also calls for generalizations. A first direction consists in using the results to solve control problems, for instance the problem of temporal constraints for max-plus linear systems, that we already mentioned, and was treated in particular cases in [7, 11, 12]. This is useful in production management. To implement the proposed control law is required the knowledge of the state vector $x(k)$, including its initial value $x(0)$. If the state is not directly measured, its reconstruction may be needed. A second direction is to treat in a similar way the concept of conditional invariance, that is useful for observation purposes, to reconstruct the state online, and gave rise to first discussions for systems over rings [9] and systems over semirings [10].

References

[1] G. Basile, and G. Marro, *Controlled and conditioned invariant subspaces in linear system theory*, J. Opt. Th. & Appl., Vol. 33, 1969, pp. 306-315.

[2] W. M. Wonham, A. S. Morse, *Decoupling and pole assignment in linear multivariable systems: a geometric approach*, SIAM J. Control , Vol. 8, 1970, pp. 1-18.

[3] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 3rd ed. New York: Springer-Verlag, 1985.

[4] G. Basile, and G. Marro, *Controlled and Conditioned Invariant in Linear System Theory*, Prentice Hall, Englewood Cliffs, New York, 1992.

[5] M. L. J. Hautus, *Controlled invariance in systems over ring*, in: D. Hinrichsen, and A. Isidori (Eds.), Proceeding of the Joint Workshop on Feedback and Synthesis of Linear and Nonlinear Systems, Lecture Notes in Control and information Sci., vol 39, Springer, New York,1982, pp. 107-122.

[6] G. Conte, and A. M. Perdon, *The disturbance decoupling problem for systems over ring*, SIAM J. Control Optim., Vol. 33, 1995, pp. 750-764.

[7] R. Katz, *Max-plus (A, B)-invariant spaces and control of timed discrete-event systems*, IEEE Trans. Automatic Control, Vol. 52, 2007, pp. 229-241.

[8] N. Ito, and H. Inaba, *Dynamic feedback (A,B)-invariance of submodules for linear systems over commutative Noetherian domains* , Lin. Algebra Appl. 282 , 1998, pp. 123-129.

[9] M. Di Loreto, J.-F. Lafay, and J. J. Loiseau, *Some Remarks on Duality over a Commutatoive Ring*, Math. and Computers in Simulation, Vol. 76, 2008, pp. 375-387.

[10] M. Di Loreto, S. Gaubert, R. D. Katz, and J. J. Loiseau, Duality between invariant spaces for max-plus linear discrete event systems, SIAM J. Control Optim., Vol. 48, 2010, pp. 5606-5628.

[11] C. Maia, C. R. Andrade, L. Hardouin, *On the control of max-plus linear systems subject to state restriction*, Automatica, Vol. 47, 2011, pp. 988-992.

[12] S. Amari, I. Demongodin, J. J. Loiseau, and C. Martinez, *Max-plus control design for temporal constraints meeting in timed event graphs*, IEEE Trans. Automatic Control, Vol. 57, 2012, pp. 462-467.

[13] G. L. Litvinov, and S. N. Sergeev (Eds), *Tropical and idempotent mathematics and applications*, Series Contemporary Mathematics, Vol. 616, AMS, 2014.